DYNAMICS OF PLANET-CROSSING ASTEROIDS

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Abstract. A small, but by no means negligible, fraction of the small bodies of the Solar System is on planet-crossing orbits, including Earth-crossing ones. The dynamics of planet-crossing asteroids/comets is strongly controlled by the occurrence of close approaches. The node crossing cycle, resulting from the secular evolution of the orbital elements, especially the argument of perihelion, is apparent in the evolution of all the elements, including the semimajor axis. The most common type of orbits defines the Geographos class, in which close approaches occur at random whenever they are made possible by the distance of the orbits. Other orbits are protected from close approaches either by mean motion resonances (Toro class) or by secular perturbations (Kozai class). The Alinda class is defined by the presence of a mean motion resonance with Jupiter, which can change over a comparatively short time the eccentricity and therefore the crossing behavior. In all cases the orbits are chaotic, and in the long run transitions between the different orbit types can occur. This paper summarizes the experimental evidence, resulting from numerical integrations, and the semianalytical theories (based upon the adiabatic invariant and upon the Kozai approximation) which can explain in a satisfactory way most of the dynamical behaviors found in the experiments.

1. THE ASTEROID COMPLEX, NEAR EARTH OBJECTS

The asteroids are a population of small bodies, whose distribution in space is concentrated in a *main belt* between the orbits of Mars and Jupiter (Figure 1). A significant number of objects is also found in the two Trojan clouds, roughly at the same distance from the Sun as Jupiter. A compara-

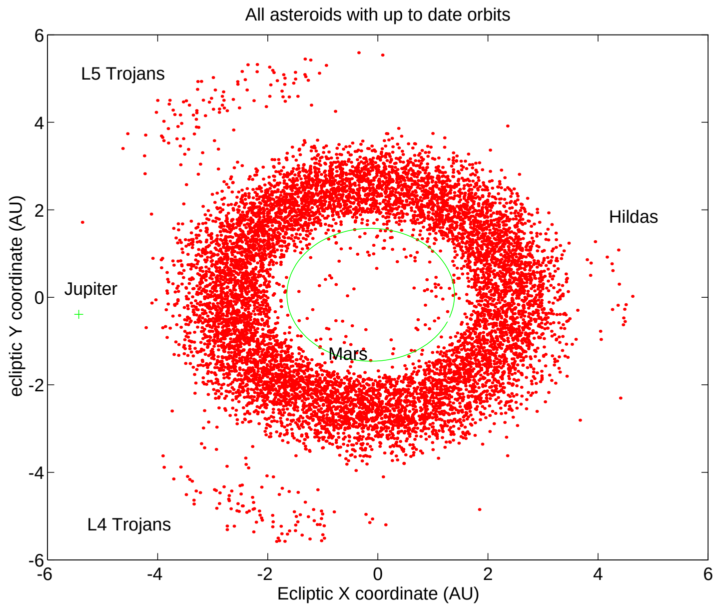

Figure 1. The asteroid complex projected on the ecliptic plane; the x, y coordinates are in astronomical units (AU). Most asteroids are found between the orbits of Jupiter (whose current position is the cross at the left) and the orbit of Mars (the line near the center), but a small fraction of the asteroids are planet-crossers.

tively small fraction of this population has orbits which can cross the orbit of some major planet; this is apparent from a plot, such as Figure 2, showing only asteroids down to given size. However the number of asteroids grows in a steep way as the minimum size considered decreases; thus at smaller sizes a comparatively large number of asteroid orbits with perihelia closer to the Sun than Mars, and even the Earth and Venus, have been discovered.

Figure 3 shows the orbital elements (a, e) of these smaller asteroids, and also of the short periodic comets, which are bodies of sizes comparable to the asteroids but more easily visible because of the release of gas and dust; these comets have often orbits crossing the orbit of Jupiter. A figure such as this one, showing the orbital elements of many small asteroids, gives a good intuitive understanding of the transport routes between the main belt and the Earth neighbourhood: these routes appear as trails in Figure 3; as an example, one route exits the main belt through the $3:1$ mean motion resonance at $a \simeq 2.5\,AU$, another one exploits the secular resonance around $a \simeq 2.2\,AU$.

There is a very large literature on the dynamics of main belt asteroids (also of Trojans; see [17] and references therein); the dynamical behavior of planet-crossing orbits has been studied only in comparatively recent times,

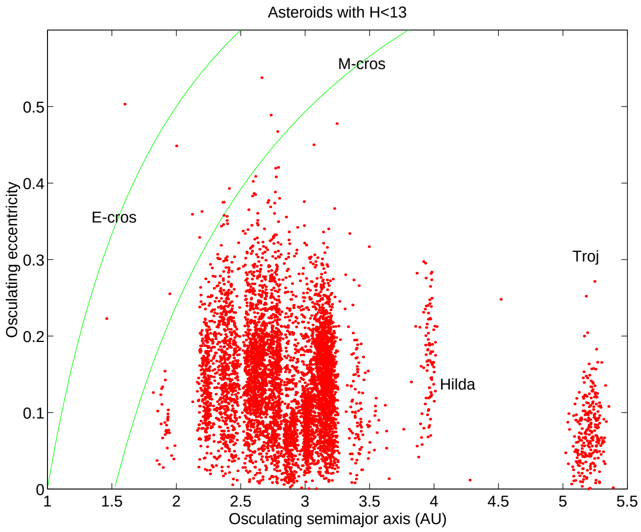

Figure 2. The asteroid complex represented in the semimajor axis a and eccentricity e plane; most asteroids are *main belt*: their orbits do not cross the orbits of Jupiter and Mars. The Trojans cross the orbit of Jupiter, very few asteroids brighter than absolute magnitude 13 ($\simeq 10\,km$ in diameter) are beyond the Mars-crossing and the Earth crossing lines.

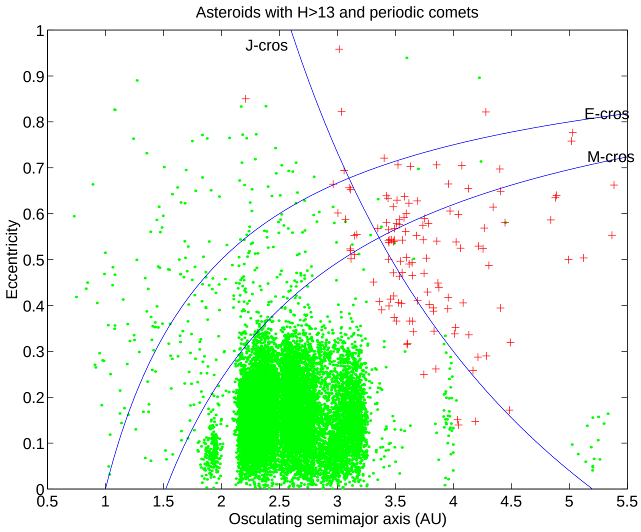

Figure 3. The asteroids smaller than absolute magnitude 13 (dots) show a distribution, in the same a, e plane of Figure 2, extending well beyond the Mars-crossing line and even the Earth-crossing line. The short periodic comets (crosses), that is the minor planets which have, at least at some apparition, detectable emissions of gas and dust, are often found in Jupiter-crossing orbits.

and most of the studies are only descriptions of the output of numerical experiments. Clearly the numerical integration of a large sample of orbits is a necessary but not sufficient condition for understanding the complex problem of planet-crossing dynamics: the familiarity with a large set of examples needs to be supported by a number of theoretical tools. The purpose of this paper is to introduce at least some of the examples of dynamical behavior and some of the conceptual tools to understand them.

2. NORMAL PLANET-CROSSING: GEOGRAPHOS CLASS

Let us consider a planet, e.g. the Earth, on a roughly circular orbit with radius a', and an asteroid with an orbit having the perihelion distance $q = a(1-e) < a'$ and aphelion distance $Q = a(1+e) > a'$. Under these conditions it is clear that the two orbits could have a common point, but we need to consider the geometry in 3 dimensions.

Each of the two osculating orbits lies in a plane; these two planes being, in general, distinct, have a common line, the *line of nodes*. The two points of the asteroid orbit on the orbit plane of the Earth are the *ascending* and the *descending node*, with distances from the sun:

$$r_+ = \frac{a(1-e^2)}{1+e\cos\omega} \quad ; \quad r_- = \frac{a(1-e^2)}{1-e\cos\omega}$$

with a minimum of q and a maximum of Q, depending upon the value of the *argument of perihelion* ω. For some values of ω a *node crossing* occurs, when one of the two *nodal distances* $r_+ - a'$ and $r_- - a'$ is zero, and one of the two nodes of the asteroid orbit belongs to the osculating orbit of the Earth.

In practice, the orbital elements of the asteroid orbit are by no means constant, they slowly change, mostly as a result of *secular perturbations.* Over time scales of $10,000 \simeq 100,000$ years, in many cases, the eccentricity e and inclination I undergo small relative changes, while the argument of perihelion ω performs several revolutions. In this case the asteroid is a *quadruple crosser*, that is there are four times during each period of circulation of ω when node crossings occur, one for each quadrant (Figure 4). In other cases, the changes in e, I are not negligible, and more complicated sequences of node crossings can occur, with up to 8 node crossings per period of ω.

The existence of node crossing has three main implications. First, the classical analytical theories to compute short periodic perturbations, secular perturbations, resonances, and solutions of any kind, are either not available or do not work properly. This results from the presence of the singularity of collision, which can occur if the Earth and the asteroid are

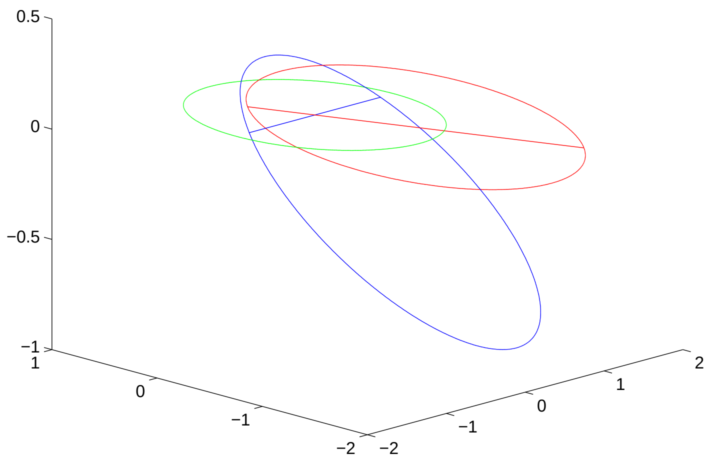

Figure 4. Node crossing: the two elliptic orbits in this plot have the same semimajor axis a, eccentricity e and inclination I, but different values of the longitude of the node Ω and of the argument of perihelion ω; the circular orbit has a semimajor axis a' such that $a(1-e) < a' < a(1+e)$. For $\omega = 0$ the elliptic orbit is linked to the circular orbit (like two consecutive links in a chain); for $\omega = \pi/2$ the ellipse and the circle are unlinked, thus for some intermediate value of ω there has been a crossing of the two orbits along the line of mutual nodes.

passing through the common node at the same time. The classical perturbative theories use either series expansion, which are in this case divergent, or averaging, which results in improper (often divergent) integrals.

Second, the possibility of close approaches of the asteroid to the Earth (also to other planets) results in chaotic motion. The correlation between the Lyapounov time, over which two nearby orbits increase their distance on average by a factor exp(1), and the average time between close approaches, is well understood and has been tested in a quantitative way by [30] and by [27]. To explain in a simple way this phenomenon we can resort to a piecewise 2-body model: during the time span when the nodal distance is small enough for close approaches, if the phases of the Earth and of the asteroid along their respective orbits are independent, there is a finite chance that an encounter occurs. When this happens, we use a 2-body asteroid–Earth model, in which the asteroid follows an hyperbolic orbit around the Earth (Figure 5). A bundle of nearby orbits, with changing impact parameter with respect to the Earth, gets scattered with distances increased by a factor of 2 to 3, if the closest approach is inside $0.1\,AU$. This increase of the mutual distances of nearby orbits is repeated at each occurrence of a close approach, and each time the factors of increase multiply the previous

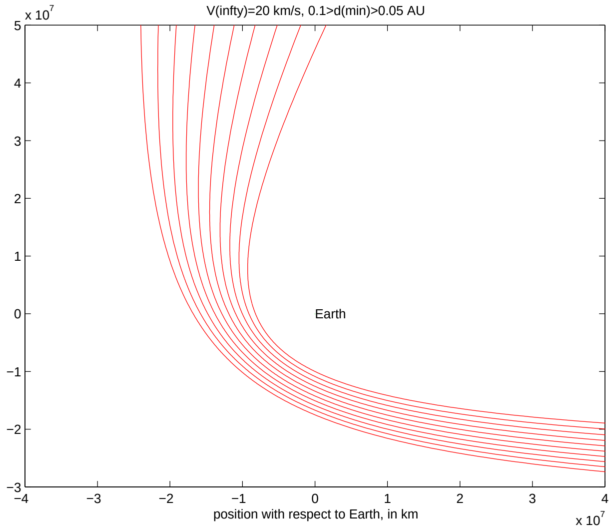

Figure 5. The increase in distance between nearby orbits during a close approach is illustrated by these hyperbolic orbits around the Earth. In this example, the asymptotic velocity at infinity is $20\,km/s$ and the minimum distance between 0.05 and $0.1\,AU$.

increase; this generates an exponential divergence, with Lyapounov time roughly equal to the average time span between two close approaches.

As it is well known from both theory and numerical experiments, chaotic orbits exhibit random behavior, alternating between different states, and also diffusion in the phase space which could result in large relative changes of the orbital elements; all the planet-crossing region of the phase space is indeed a single large chaotic sea, within which every transition if possible, although not necessarily probable. Typically cometary orbits and typical near Earth asteroid orbits can be connected by an evolutionary path including deep close approaches; a good example is the numerically computed orbit of *(1862) Apollo* shown in [20], Figures 5 and 6.

Third, the occurrence of node crossings only for some values of the argument of perihelion, which has "secular" periods of many thousands of years, results in intermittent behavior. When the nodal distance is small, frequent close approaches result in a random walk of all the orbital elements, even the semimajor axis, with occasional large jumps. When on the contrary the minimum distance between the two orbits is high (say $> 0.2\,AU$), the asteroid orbit shows all the features of a regular orbit, which could be represented by a Fourier series, over which averaging is meaningful, for which proper elements could be computed. However, this regular behavior only lasts until the next node crossing, when the orbital elements get random walked to some other value, from which a new regular orbit segment can begin.

Thus it is very difficult to make use of any statistical representation of the chaotic motion of an Earth-approaching body. If some kind of ergodic principle does hold for this class of orbits, that is if it is possible to exchange averages over regions of the phase space with time averages, this can occur only over times many orders of magnitude longer than the time span between two node crossings; these time spans are in practice too long to be explored numerically, even with the fastest computers of today, unless approximations are introduced which can not represent in a reliable way the real behavior of the exact orbits.

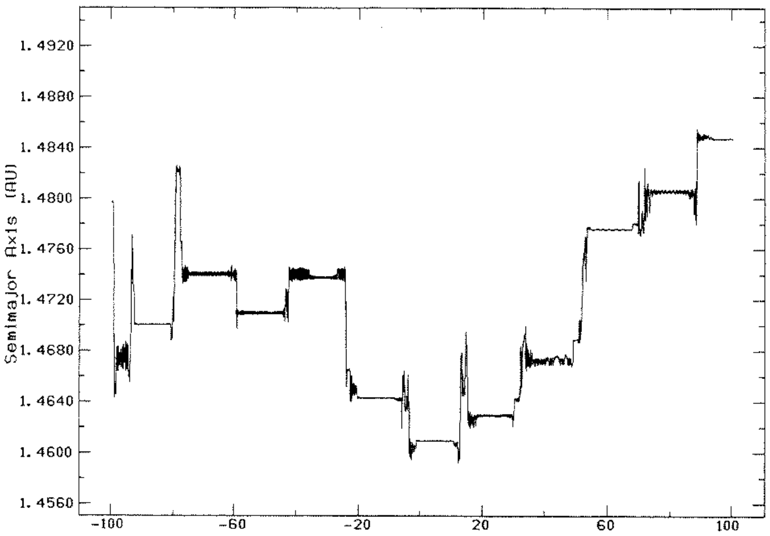

Figure 6. The semimajor axis of *(1864) Daedalus* as a function of time, from the output of the numerical integrations of *PROJECT SPACEGUARD*, over a time span of 200, 000 years. The time is is marked on the horizontal axis in thousands of years, in this as well as in the following figures. Jumps occur at node crossings with the Earth (see Figure 7) and are the result of close approaches (see Figure 8); when the nodal distance is comparatively large, the semimajor axis undergoes only small oscillations, and has negligible secular variations.

The orbits which show this intermittent behavior, alternating between random occurrence of close approaches near node crossings and regular evolution of the orbital elements (with a almost constant) when nodes are far apart, belong to the *Geographos* class, according to the classification proposed by [20]. This classification is based on the analysis of a set of 89 Earth-crossing orbits numerically computed for 200,000 years; the Geographos class was the most populated one in that sample, which included orbits corresponding –although only approximately– to all the Near-Earth Asteroids known at the time. Thus we can say that the Geographos are

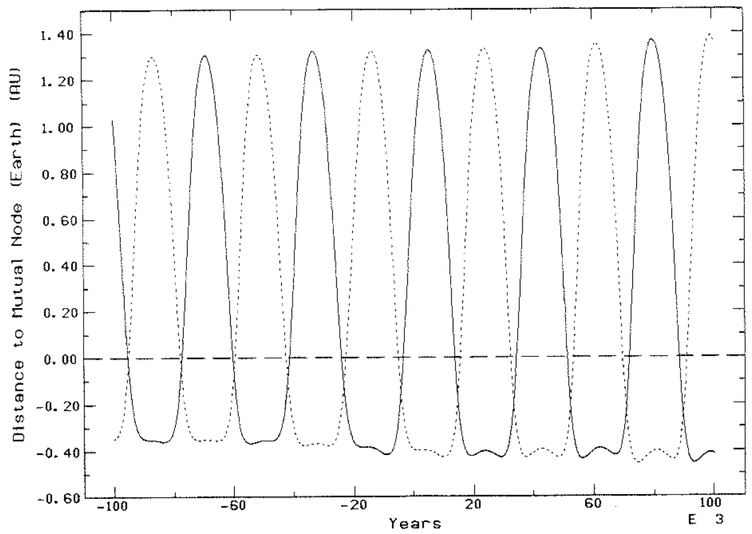

Figure 7. Nodal distance of the asteroid *(1864) Daedalus* with the Earth, as a function of time; the continuous line refers to the ascending node, the dotted one to the descending node. Four node crossings occur for every revolution of the argument of perihelion ω.

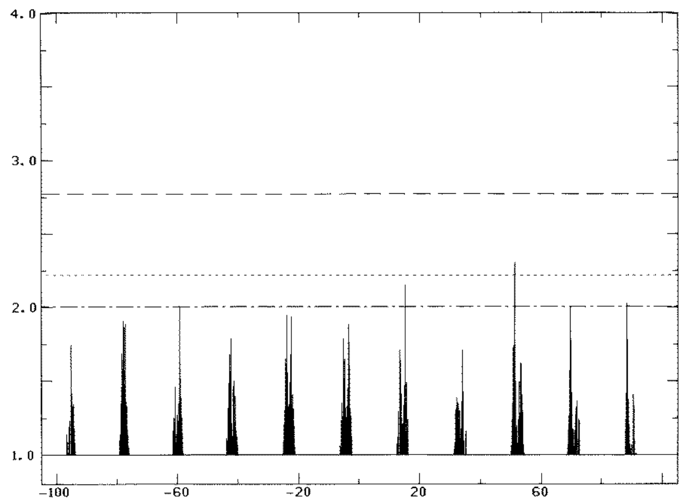

Figure 8. Close approaches of *(1864) Daedalus* to the Earth; in this *comb plot* each black line represents a close approach, and the height is the base 10 logarithm of the inverse of the minimum distance (in AU). By comparing with Figure 7, it is clear that close approaches occur at random whenever they are made possible by the distance of the orbits.

the "normal" Earth-crossing orbits, although their dynamical behavior is abnormal enough when compared to the one of the main belt asteroids.

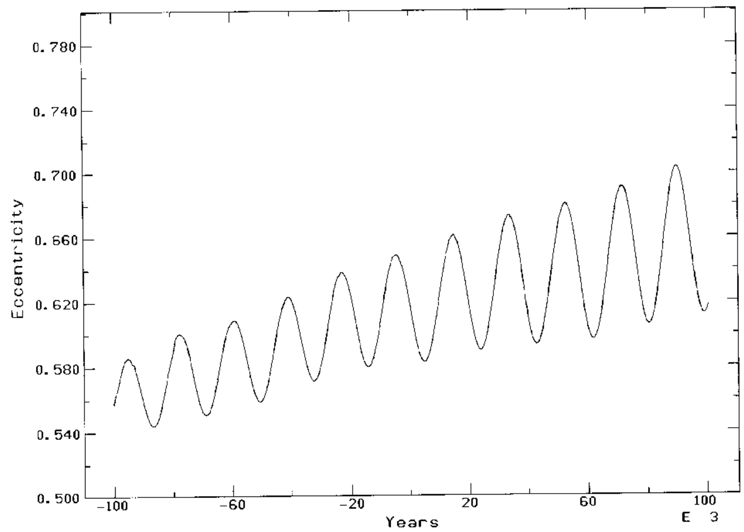

Figure 9. The eccentricity of *(1864) Daedalus* undergoes smooth long term perturbations, which do not appear to be strongly affected by the close approaches occurring near the node crossings. In a sense to be defined, the secular perturbations exist and control the evolution of the elements different from the semimajor axis; in this case, a secular increase in eccentricity is the most important dynamical evolution taking place.

As an example, Figures 6-9 show the behavior of *(1864) Daedalus*, a typical Geographos class asteroid. The semimajor axis (Figure 6) is almost constant over time spans of several thousand years, then jumps to significantly different values, in such a way that on the long term it undergoes a kind of random walk. The nodal distance (Figure 7) crosses the zero lines near the times of the jumps in a; the actual occurrence of close approaches follows an intermittent pattern (Figure 8), which can be described as random occurrence of encounters whenever the distance between the two orbits is small enough. The eccentricity (Figure 9) changes in an almost smooth way, as if secular perturbations exist in some sense and control the evolution of the orbital elements e, ω; the same is true for I, Ω.

Two main questions arise from this empirical description of the dynamical behavior of most Earth-crossing orbits. The first question is whether it is possible to exploit this alternating, but in some sense qualitatively uniform, dynamical behavior of the Geographos class orbits to give some statistical description, as an example to compute the probability of collision with the Earth. The second question is whether it is possible to develop an analytical (or at least semianalytical) theory to account for the ostensibly regular behavior of the orbital elements e, I, ω, Ω even in the presence of node crossings and close approaches.

3. STATISTICAL THEORIES FOR ENCOUNTERS

The attempts to model the occurrence of both close approaches and collisions for planet–crossing orbits have been based upon a statistical approach, starting from the pioneering effort by Öpik, see [25]. A statistical theory can be constructed by adopting three simplifying assumptions:
[1] The orbits between two consecutive close approaches are assumed to be regular and modeled in some simplified way, such that an explicit computation is possible. In the simplest possible approach, the orbits are modeled as keplerian ellipses with constant semimajor axis a, eccentricity e and inclination I, uniformly precessing with time; more elaborate models exploit secular perturbation theories, obtained by either analytical or semianalytical methods, in which the constants of the motion are "proper" a, e and I as opposed to the corresponding instantaneous, or "osculating", elements.
[2] Close approaches and collisions are modeled as purely random events. More specifically, whenever the keplerian ellipses representing the osculating orbits of one asteroid and one planet are close enough to allow for a close approach, the position of the planet and of the asteroid on the orbit (i.e. the anomalies) are assumed to be randomly distributed and uncorrelated; mean motion resonances, which constrain the two anomalies, are not allowed in the model. Each close approach is seen as a random event uncorrelated with any previous one.
[3] The orbital elements $a\,, e,\, I$ of the asteroid are assumed to change only as a result of close approaches; in the more refined theories, this applies to the proper elements. Since at deep close approaches these changes can be computed e.g. by a hyperbolic two–body encounter approximation, the orbital evolution can be described as a random walk. For each present state, the probability (hence the expected waiting time) for a given change in the elements can be explicitly computed; this makes possible the use of Monte Carlo simulations. In some more sophisticated theories, other events which can result in changes of the elements, such as the capture in some resonance, are taken into account.

The simplest theories in this class, such as the one by Kessler, estimate the frequency of the occurrence of close approaches by modeling the asteroid orbit as if it were a probability cloud. By averaging over the angular variables ℓ (mean anomaly), ω and Ω, the probability density of the asteroid position is a function of a, e, I; because of the averaging over Ω, the probability is independent from longitude. The resulting density $S(r, \beta)$ at a radius r from the Sun and at an ecliptic latitude β is given by [10], [11]:

$$S(r,\beta) = \frac{1}{2\pi^3 r a \sqrt{(\sin^2 I - \sin^2\beta)(r-q)(Q-r)}}$$

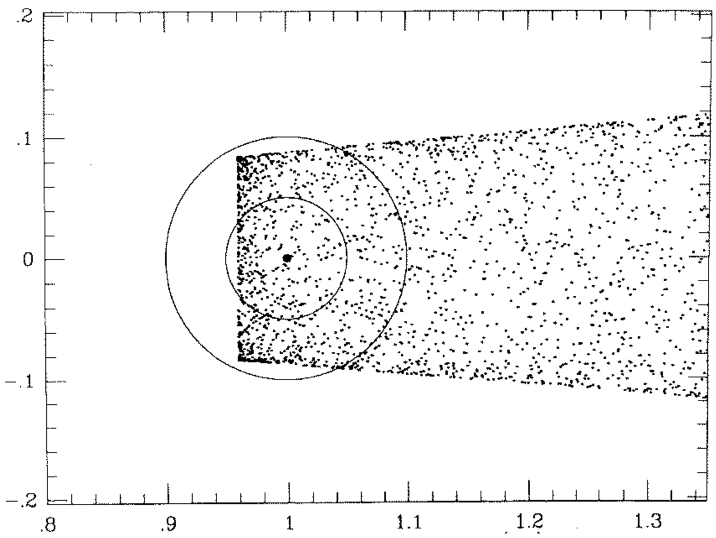

Figure 10. The probability density of the asteroid position has been simulated by a random sample of points on orbits with random values of ℓ, ω, Ω; here they are plotted on the r, β plane (radius from the sun in AU, on the horizontal axis; ecliptic latitude in radians, on the vertical one). In the simplest Kessler type formula, the number of encounters within an impact parameter D is estimated by multiplying the probability density at the position of the Earth $\beta = 0, r = 1$ by the cross section πD^2. A more sophisticated formula takes into account the changes of the probability density with the position across the cross section disk, which is especially important when the cross section disk contains part of the boundary of the region supporting the probability cloud, as in the two cases shown in this plot.

provided r is larger than $q = a(1-e)$ and smaller than $Q = a(1+e)$, and $|\beta|$ is smaller than I; S is zero if any of these constraints is not satisfied. Then the most likely number of close approaches per unit time through a cross section A can be computed from S and from the relative velocity V of the two orbits:

$$P(r,\beta) = S(r,\beta) V A$$

The above formula needs to be averaged over the target orbit; in the approximation of a circular orbit for the target planet, with $e' = I' = 0$and with $r = a'$ the semimajor axis of the planet:

$$P = \frac{VA}{2\pi^3 a a' \sin I \sqrt{(a'-q)(Q-a')}}$$

where V can be computed from the known value of the velocity of the asteroid orbit when node crossing occurs.

The cross section A for a close approach within an impact parameter smaller than D is just πD^2, and the relationship between impact parameter and minimum distance d_{MIN} is defined by the two–body hyperbolic encounter formula:

$$D^2 = \left(1 + \frac{2GM'}{d_{MIN}V^2}\right) d_{MIN}^2$$

with GM' the gravitational constant of the planet.

More complicated formulas have been derived to take into account the eccentricity e' of the orbit of the planet [26]; but the main problem is not in the approximation $e' = 0$. For either $r = q$ or $r = Q$ or $\beta = \pm I$ the probability density is singular (infinite); nevertheless the probability of collision is finite, when computed as an improper integral over the cross section. Thus it is possible to regularise the apparent singularity of the probability density, e.g. with the semianalytic method of [21].

In practice, the orbits with close approaches near either perihelion ($q \simeq a'$) or aphelion ($Q \simeq a'$), and/or with very small I can have a very large number of close encounters, and the asteroids whose orbits evolve through one such state have probabilities of collision comparatively high and difficult to compute exactly (because of the instability of the computation). If these difficult cases are handled with care, it is possible to achieve a good accuracy in the prediction of the deep close approach frequency for a large population of orbits.

As an example, in [21] the average probability of collision with the Earth for a Near Earth orbit (with $q < 1\,AU$ at least for some time) is estimated by extrapolating from the sample of close approaches found in a $200,000$ years integration at 3.9×10^{-9} per year per object. The same computation done on the basis of a modified Kessler theory (with regularisation) gives a probability of 3.6×10^{-9} per year per object. Other authors give not very different results [**?**]. This means that the probabilities of collision, when averaged over a long time span and over a large number of orbits, can be computed in sufficiently accurate way. The main source of uncertainty on the probability of collision is in our very incomplete knowledge of the population of planet crossing objects, and does not depend upon lack of mathematical knowledge.

4. THE PROTECTED ONES: THE TORO CLASS

An Earth-crossing orbit of the "normal" type, that is of the Geographos class, undergoes close approaches at random, during comparatively short time spans around the epochs of the node crossings. However, there are orbits which are Near Earth in that they satisfy $q < 1\,AU < Q$, and nevertheless close approaches either do not take place at all, or take place more seldom and with larger distances from the Earth than the statistical theories would predict.

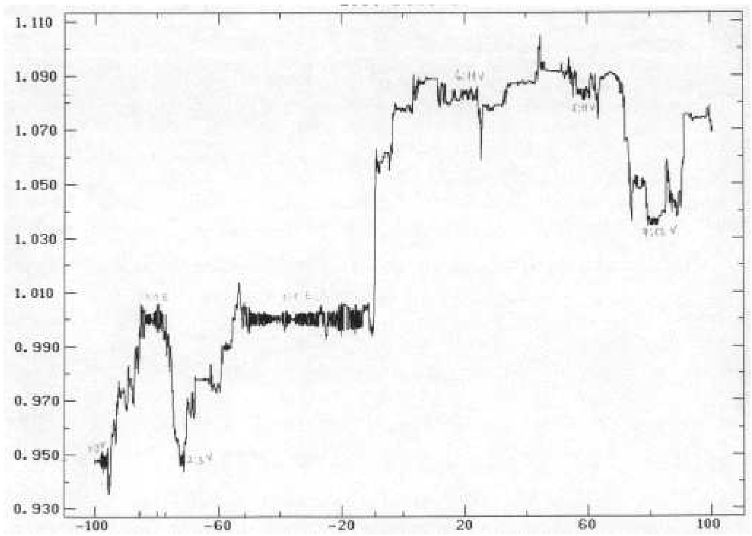

Figure 11. Semimajor axis of the asteroid *(2063) Bacchus*, as a function of time, from the output of the numerical integrations of *PROJECT SPACEGUARD*, over a time span of 200, 000 years. A transition between an *Aten* type orbit (with $a < 1\,AU$) and an *Apollo* type (with $a > 1\,AU$) occurs at $\simeq -10,000$ years from present time, as a result of a very close approach ($\simeq 2$ radii of the planet) with Venus.

The *protection mechanisms* which can be responsible for this "Earth-avoidance" behavior are essentially two: either node crossings occur, but close approaches do not take place even when the distances between the two orbits would allow, or node crossings do not occur. In this Section we discuss the first case, that is the *Toro class*, according to the classification of [20].

The empirical evidence for the existence of a Toro class of dynamical behavior is in a number of examples found by several authors, starting from [9], and in a systematic way in the large database of planet-crossing orbits generated by the *PROJECT SPACEGUARD*, which contain thousands of figures like Figures 11-13. The Figures we have selected refer to the asteroid *(2063) Bacchus*, and show the 'normal' Geographos class behavior, with semimajor axis jumping at random as a result of close approaches to both the Earth and Venus (Figure 11) around the node crossing epochs (e.g. Figure 12 shows the node crossings with the Earth). However, there are some node crossings with the Earth which do not correspond to close approaches: in the Figure 13 these "safe intervals" are apparent as gaps in the comb. In this example the protection mechanism is active when the semimajor axis of *(2063) Bacchus* is close to $1\,AU$, and therefore must be related to the $1:1$ resonance, that is to a state of "Earth's Trojan".

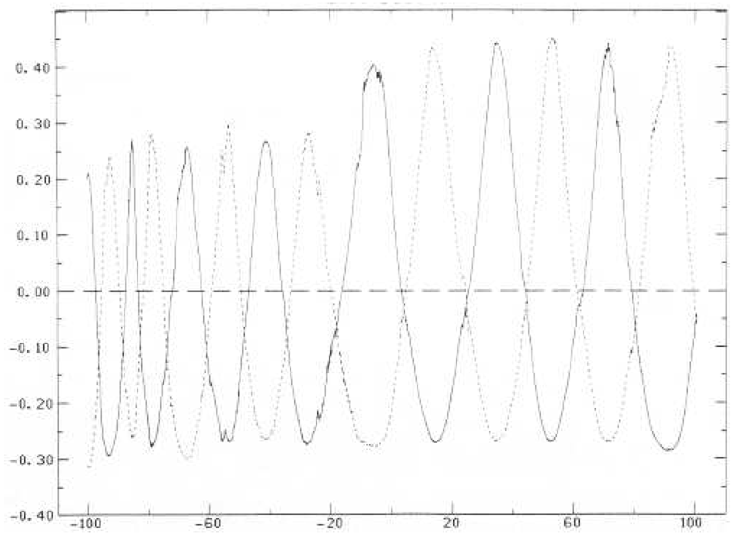

Figure 12. Nodal distance of the asteroid *(2063) Bacchus* with the Earth; the continuous line refers to the ascending node, the dotted one to the descending node. The node crossing are comparatively frequent, and when the nodal distance with respect to the Earth is close to $-0.3\,AU$, almost tangent encounters with Venus can take place.

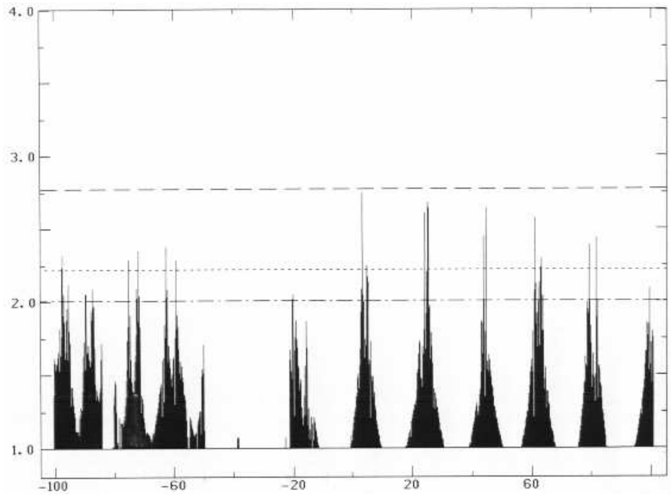

Figure 13. Close approaches of *(2063) Bacchus* to the Earth; in this *comb plot* each black line represents a close approach. By comparing with Figure 11, it is clear that close approaches do not occur near eight node crossings: two around $-80,000$ years, and a sequence of six between $-50,000$ and $-20,000$ years.

To model in the simplest way this behavior, let us assume the planet –e.g. the Earth– is in a circular orbit with radius a', and the asteroid has or-

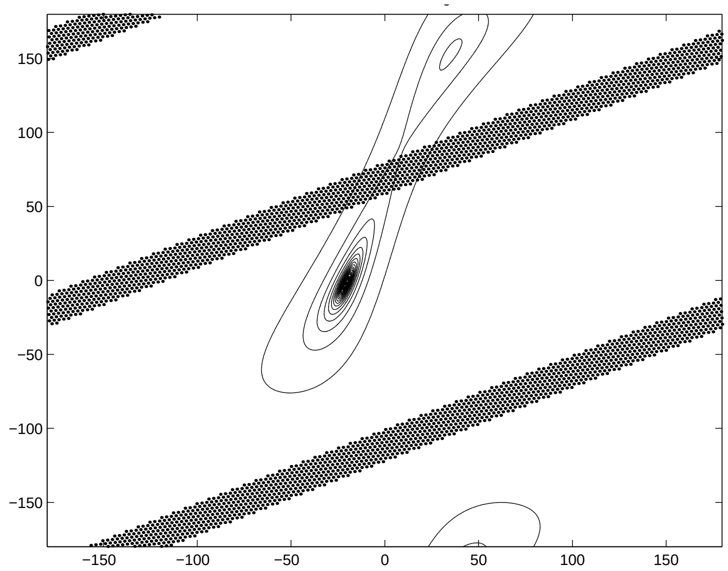

Figure 14. Level curves of the inverse distance between the asteroid and the Earth –that is, apart from a constant factor, the direct perturbing function– as a function of the two mean anomalies. The perturbing function has a very sharp maximum near the ascending node, with both ℓ and ℓ' near $0°$, because the orbital elements (a, e, I, ω) are such that the ascending node is near $1\,AU$. On the same plot, the mean anomalies evolving linearly with time, with a ratio of mean motions close to $1:2$, and a phase such that the region of close approaches is avoided.

bital elements (a, e, I, ω) such that –for some time around the node crossing epoch– the distance from the Sun at one node is close to a': let us say

$$r_+ = \frac{a(1-e^2)}{1+e\cos\omega} \simeq a' \ ;$$

note that the value of the longitude of the node Ω does not matter. For a time span short with respect to the frequencies of the secular perturbations, but longer than the orbital period, the only elements which change in a significant way are the anomalies ℓ, ℓ' of both the asteroid and the planet. Since these are angle variables, the short term dynamics can be described on a phase space which is a *torus*, the Cartesian product of two circles; a torus can be represented, as in Figure 14, as a square with the opposite sides identified.

Figure 14 shows the level lines of the gravitational potential of the Earth, as felt by an asteroid in an Earth-crossing orbit very close to a node crossing (the distance at the ascending node is less than $0.1\,AU$). The gravitational potential has a very sharp maximum near the node, while the perturbation due to the Earth is very small elsewhere. As time goes by, the changes in the anomalies ℓ, ℓ' of both orbits can be described approximately by the linear functions of time $\ell = nt + \ell_0$; $\ell' = n't + \ell'_0$, with n, n' the mean motions. For values of a such that the ratio n/n' is close to a fraction q/p with q, p small integers, that is near a *mean motion resonance*, the orbit does not spread uniformly on the torus, but can avoid significant portions of it, possibly including the region around the maximum, that is avoiding close approaches.

That such a temporary protection mechanism can occur at some node crossings is clear, but the question is: does this happen by chance? By means of a purely cinematical description, as given above, we would expect that if a is close enough to a resonant value, the orbit will have the right phase ℓ_0 for protection at some node crossing, and then have a wrong phase in some later node crossing, essentially at random. This is not the case because the mean anomaly of the asteroid, and therefore the phase of the resonance (some *critical argument* of the form $p\ell - q\ell'$+some combination of $\omega, \Omega, \omega', \Omega'$), does not change linearly but is subject to a kind of restoring force. The restoring force is of course stronger the closer are the two orbits, and has very high values near the node crossings, and near the maximum point on the torus.

The behavior of the critical argument of a Toro class orbit is shown as a function of time in Figure 15, from [19]. The plot shows “avoidance”, in the sense that the dangerous position with both the Earth and the asteroid near the ascending node is avoided, but only when the nodal distance is small (less than $0.13\,AU$ in this example). This Figure is typical of the Toro class orbits, including the case of the namesake asteroid *(1685) Toro*, Figure 6 in [20].

Thus the protective effect of mean motion resonance occurs much more often than it would take place if it was controlled only by chance. Figure 15 is enough to understand that the restoring force due to the perturbations from the Earth acting on the asteroid mean anomaly ℓ is pushing the orbit away from the collision.

This “negative attraction” effect is somewhat counter our earth-bound intuition, but in fact is another form of the first paradox of astrodynamics, well known to astronauts. If an asteroid (or spacecraft) is pushed forward along its orbit by some perturbing acceleration acting along track (it does not matter if this acceleration is due either to the attraction of a third body, or to the action of rocket engines), the orbital energy increases, the semimajor axis increases, and the mean motion decreases, thus the asteroid (spacecraft) is pushed backward; the displacement produced by an along track acceleration Δf is $\Delta x = -3/2\, t^2\, \Delta f$.

To transform this intuitive explanation in a rigorous mathematical ar-

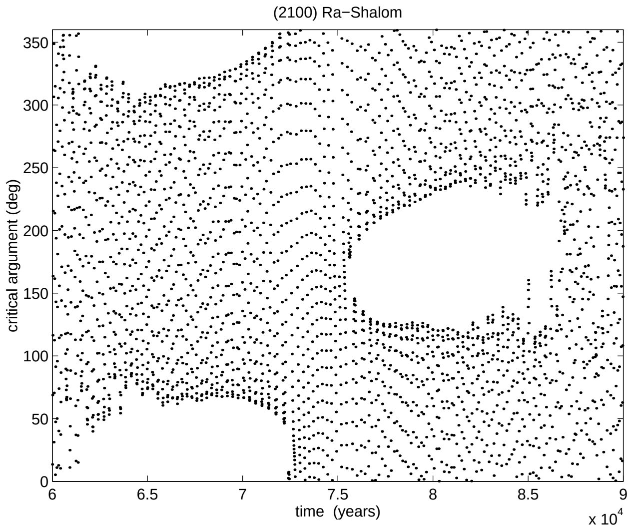

Figure 15. Critical argument $-5\lambda + 7\lambda' - 2\varpi$ (in degrees, between 0° and 360°), over the $100,000$ years time span of numerical integration, for the asteroid *(2100) Ra-Shalom*. The two gaps occurring around 0° between $\simeq 62,000$ and $72,000$ years, and around 180° between $\simeq 76,000$ and $86,500$ years, indicate the avoidance of the closest approach state by means of some protection mechanism. On the contrary, between $\simeq 72,000$ and $76,000$ years the critical argument circulates, and the semimajor axis does not undergo resonant perturbations. Node crossings occur at $\simeq 66,000$ and at $\simeq 82,000$ years.

gument we need to resort to a semianalytical theory, obtained by averaging [19]. If $\sigma = p\ell - q\ell'$ is a slow variable, because of $n/n' \simeq q/p$, then there is another variable $\tau = c\ell - d\ell'$ obtained together with σ by a unimodular transformation, that is c, d are integers and $pd - qc = 1$. Such τ is a fast angle, and averaging with respect to τ is a good approximation by the *averaging principle* [1]; if the averaging is performed over a single angle variable, the averaging principle is a rigorous theorem, with estimate of the error done in neglecting the short periodic terms. Then we are left with the semi-averaged Hamiltonian, still depending upon one angle variable:

$$\overline{H}(\sigma, \Sigma, T) = H_0(\Sigma, T) - \frac{1}{2\pi}\int_0^{2\pi} R(\sigma, \tau, \Sigma, T)\, d\tau$$

where Σ, T are the action variables conjugate to the angle variables σ, τ, H_0 is the Hamiltonian of the 2-body unperturbed problem transformed to the new variables, and R is the usual perturbing function of the 3-body problem, also depending upon the other orbital elements, especially ω which controls the nodal distance.

For the semi-averaged Hamiltonian $\overline{H}$, the fast angle τ is a cyclic variable, hence T is an integral of motion; once the value of T is fixed by the initial conditions, H_0 is a function of Σ only. If we avoid being confused by too many changes of variables, and write the result as a function of some more usual variable, such as the Delaunay $L = k\sqrt{a}$, with $k = \sqrt{GM_\odot}$ the gravitational constant in Gauss' form,

$$\overline{H} = H_0 - \overline{R} = -\frac{k^4 M_\odot^2}{2L^2} - \frac{q}{p}\, n' L - \overline{R}(\sigma, L)$$

with $\overline{R}$ the averaged perturbing function. We can now understand the "repulsive" restoring force: the first derivative of H_0 with respect to L is $n - (q/p)n'$, zero at the exact resonance; the second derivative is always negative.

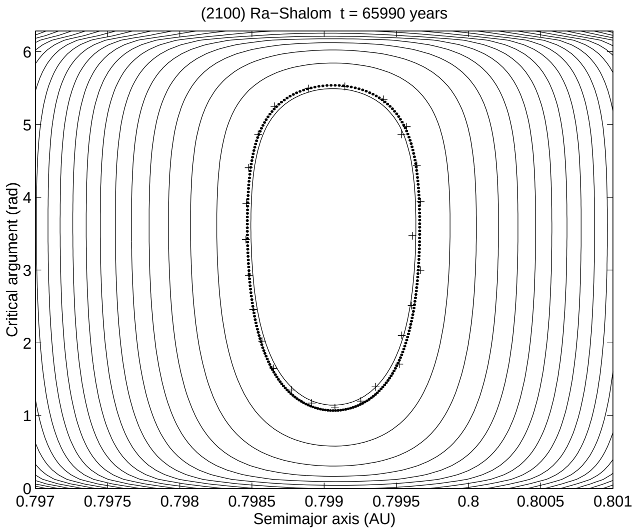

Figure 16. Level lines, in the plane with coordinates a and the critical argument (in radians), of the semi-averaged Hamilton function at the time of the node crossing; the crosses are the orbit of *(2100) Ra-Shalom*, as computed numerically by the non-averaged equations of motion. Libration of the critical argument, in this case of the $5:7$ resonance, is forced by the "infinite" potential energy maximum resulting from the averaging of the collision, in this case at the value 0° of the critical argument.

Thus the qualitative behavior of the semi-averaged Hamiltonian system can be understood by comparing with the simple Hamiltonian

$$K(x, y) = -y^2 + V(x) \ ;$$

when the nodal distance is small, $V(x)$ has a sharp minimum near the node. But the Hamiltonian is concave with respect to y, and therefore the minima of the potential energy are avoided, exactly as the maxima in

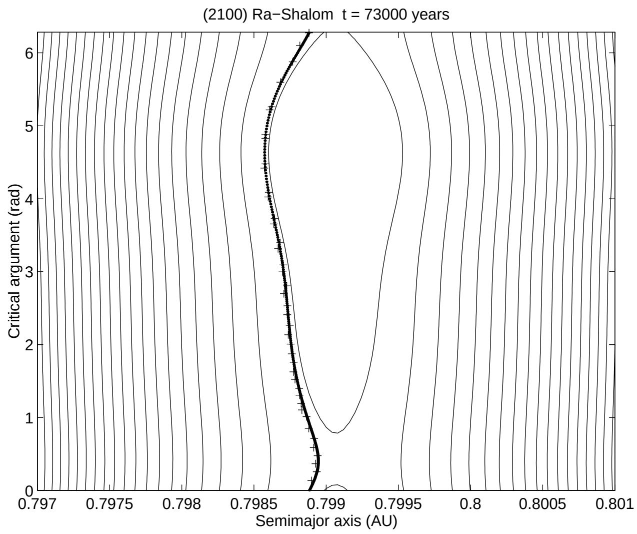

Figure 17. Level lines of the semi-averaged Hamilton function at a time when the nodal distance has increased to about $0.13\,AU$. For *(2100) Ra-Shalom* (line with crosses) the libration has changed into circulation, because the (suitably normalized) area enclosed by the curve in the previous figure is larger than the area enclosed by the separatrix in this figure.

the more familiar case with convexity with respect to y. When the nodal distance $\to 0$, the value of the minimum $\to -\infty$, and whatever the initial value of K there is an avoidance region near the singularity.

This behavior is shown in Figure 16, based on an explicit computation of $\overline{H}$ by numerical quadrature, for values of the other elements e, ω, I such that the nodal distance is very small. The changes in the elements different from a, ℓ are driven by the secular perturbations, rather than by the resonant interaction with the Earth. Thus we can consider the problem to be defined by the semi-averaged Hamiltonian $\overline{H}$, depending upon time through the slow variables e, ω, I, Ω. This is the nominal situation to apply the adiabatic invariant theory [8], by which the solutions follow closely the *guiding trajectories* provided by the solution of the system $\overline{H}$ in the (σ, Σ) plane for fixed values of the other elements. That is, there is an “almost integral”, the adiabatic invariant, which can be computed by means of the area enclosed by the level lines of $\overline{H}$ in the (σ, Σ) plane.

When the area enclosed by the separatrix curve (which is the curve through the saddle point, corresponding to the minimum distance) becomes smaller than the area required by the adiabatic invariant, then the guiding trajectory changes topology, and the solution switches from libration to circulation, as shown in Figure 17. This explains the apparently “astute” behavior of the Toro class asteroids: whenever the perturbation due to close approaches is strong, the orbit switches to a libration state, avoiding encounters at the node; when the nodal distance increases again, the potential well is not deep enough and confinement in the libration region does not occur.

The Toro state in most cases does not last for a very long time, because resonances can protect from close approaches to one planet but can not protect for a significant span of time from close approaches to two planets, for the simple reason that the planets are not resonant among them. Thus all the Toro class asteroids change their dynamical state as a result of a close approach to a planet different from the one they are protected from, typically after a time span of a few $10,000$ years. The only known exceptions are orbits which cross only the orbit of Mars, such as the Eros clones studied by [16]; if the perihelion is well above $1\,AU$, a Toro-like state with Mars can protect from close approaches, even for millions of years.

It is important to remember again that all the planet-crossing orbits are strongly chaotic, thus the long term behavior of all the orbits can not be predicted; in particular, the time span of residence of a specific orbit in a Toro state with a specific resonance is unpredictable. As an example, the time in which *(2063) Bacchus* either has been, or will be, an Earth Trojan can not be predicted, although it has to be expected that it will get there eventually.

5. THE PROTECTED ONES: THE KOZAI CLASS

The other case of ostensibly protected behavior occurs for the near-Earth orbits of the *Kozai class*, according to the classification of [20]. The name comes from a namesake asteroid, *(3040) Kozai*, but also from the author of the seminal work on this subject [12].

The evidence from numerical integrations is that of a dynamical behavior as in the example of Figures 18-19. There are many asteroids with perihelion distance q which can, as a result of the secular perturbations to the eccentricity, decrease below $1\,AU$, and nevertheless there are no node crossings, that is, the nodal distance is always positive. This happens only for orbits with significant inclination, because in this case the evolution of e and ω are strongly coupled. In the example of the Figures 18-19, that is the Mars-crossing asteroid *(1866) Sisyphus*, q can be as low as $0.8\,AU$, and at the same time the nodes are above $1.1\,AU$, because the argument of perihelion has a value either around $\pi/2$ or around $3\pi/2$ when the eccentricity is at a maximum.

To understand this mechanism of avoidance of node crossings we need to model the secular perturbations on (e, ω, I, Ω); for this we resort again

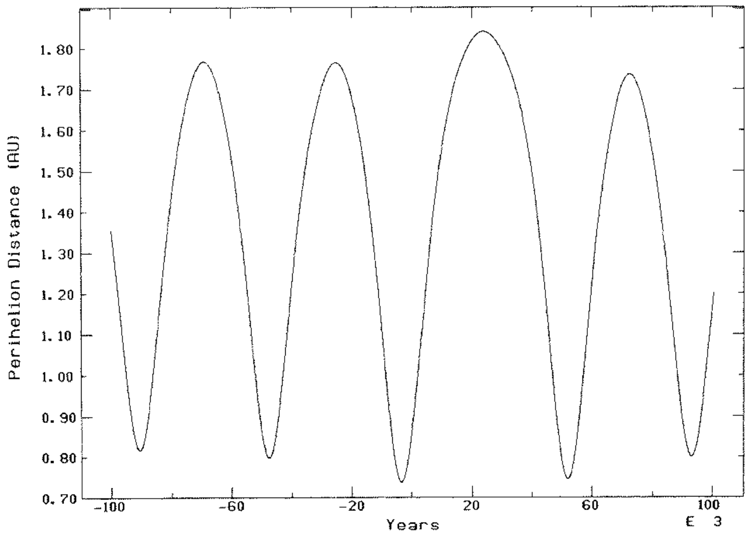

Figure 18. The perihelion distance of *(1866) Sisyphus*, when the eccentricity is near its maximum over the secular perturbation cycle, decreases well below $1\,AU$.

to the *averaging principle*, but in this case the average is performed on both the fast angles ℓ, ℓ', assuming that there is no mean motion resonance, that is the torus of the mean anomalies is in this case covered uniformly (more exactly, in an ergodic way, see [1]). The averaged Hamiltonian is

$$\overline{H} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [H_0 - R_{dir} - R_{ind}]\, d\ell d\ell'$$

where H_0 is the unperturbed 2-body Hamiltonian, $R_{dir} = k^2 m/D$ is the direct perturbing function (with D the distance between the asteroid and the planet, m the mass of the perturbing planet) and R_{ind} is the indirect part of the perturbing function. By a classical result, the average over the anomalies of the indirect perturbing function is zero, thus

$$\overline{H} = H_0 - \frac{k^2 m}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{1}{D}\, d\ell d\ell' = H_0 - \overline{R}\ .$$

There are two main methods to compute the averaged perturbing function $\overline{R}$. It is possible to expand the perturbing potential in series, with the eccentricities and inclinations as small parameters. That is, R is expanded as a Taylor series in e, I, e', I', and for each order as a Fourier series in the angles ℓ, ω, Ω. The classical D'Alembert rules, which result from the invariance of the problem with respect to orthogonal transformations, strongly constrain the terms which can appear with non zero coefficient in such a

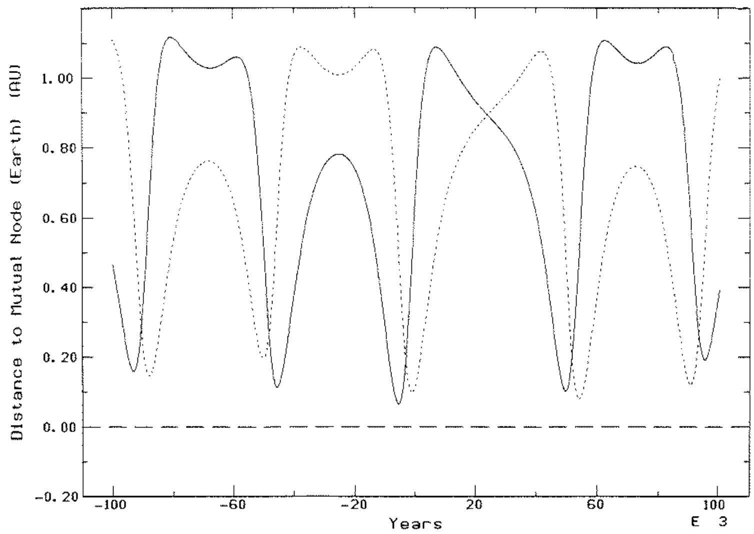

Figure 19. The nodal distance between *(1866) Sisyphus* and the Earth is always positive, that is whenever the perihelion of Sisyphus is closer than 1 *AU* to the Sun, this occurs either well above or well below the ecliptic plane.

series, and it turns out that only the even order terms are allowed; see e.g. [18]. Thus the lowest order terms containing the eccentricities and the inclinations are quadratic, and if the theory is truncated to degree two the secular perturbations can be described, in suitable variables, as the solutions of a system of linear differential equations.

The method outlined above is the classical one, introduced by Laplace and others more than 200 years ago, and it gives a reasonable first approximation of the secular perturbations of the orbits of moderately perturbed planets with low eccentricities and inclinations, such as the major planets Venus to Neptune. This approximation fails when eccentricity and inclination are large; in this case the largest term in the perturbing function neglected by the Laplace linear theory is the one with $e^2 \sin^2 I \, \cos(2\omega)$.

An integrable first approximation, different from a linear system, was introduced by [12]. Let us assume that the perturbing planet is on a circular orbit, and on the reference plane: $e' = I' = 0$. The argument applies also to the case of many perturbing planets, provided the orbits are all circular and all on the same plane. Then the problem averaged over ℓ' can be described in the way already introduced by Gauss, as the gravitational problem defined by a mass distributed in rings along the planetary orbits; the perturbing potential is axisymmetric with respect to the axis through the Sun and orthogonal to the plane of the planets, and the component of the angular momentum of the asteroid orbit along the same axis is preserved.

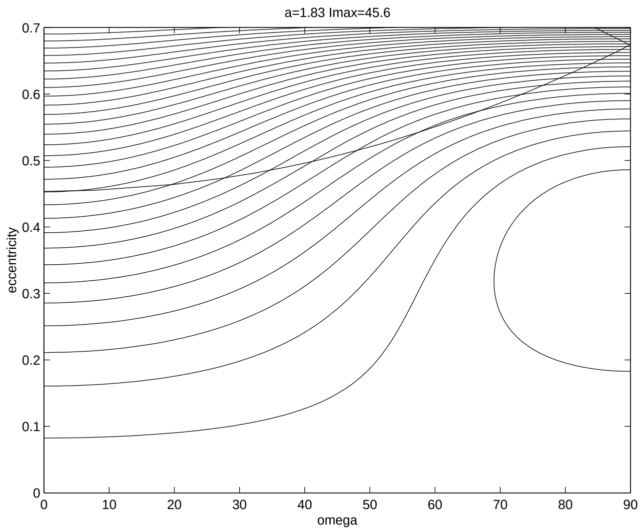

Figure 20. Level lines of the averaged perturbing function for a Kozai class orbit, in the (ω, e) plane; ω on the horizontal axis is in degrees. The node crossing lines with the Earth, where one of the two nodal distances is zero, are also drawn: the libration keeps the secular evolution of the orbits well clear of the node crossings.

In terms of $\overline{H}$ and of the keplerian orbital elements $(a, e, I, \omega, \Omega, \ell)$, the averaged Hamiltonian is independent of ℓ and of Ω, and the two conjugate variables are integrals: they are proportional to $L = k\sqrt{a}$ and to

$$Z = L\ \sqrt{(1-e^2)}\ \cos I$$

respectively. In conclusion

$$\overline{H} = \overline{H}(\omega, G; L, Z)$$

is a one degree of freedom Hamiltonian, with two parameters L and Z. This averaged problem is integrable; once a and $\sqrt{1-e^2}\ \cos I$ are fixed, the secular evolution can be described as a curve in the (e, ω) plane; I is a function of e, which can be deduced from $Z = const$, and Ω does not matter. This curve can be drawn as level curve of the function $\overline{H}(\omega, e)$.

For low inclination the $\overline{H} = const$ curves in the (e, ω) plane are not very different from $e = const$ lines; but for large I, the coupling terms such as the one with $e^2 \sin^2 I \cos(2\omega)$ become dominant, and the eccentricity undergoes large relative changes as ω circulates. Below a critical value of Z/L, the topology of the level curves of $\overline{H}$ changes and a separatrix appears, bounding a region where ω librates, typically around either $\pi/2$ or $3\pi/2$. This *Kozai resonance* is especially effective in keeping the perihelion of the

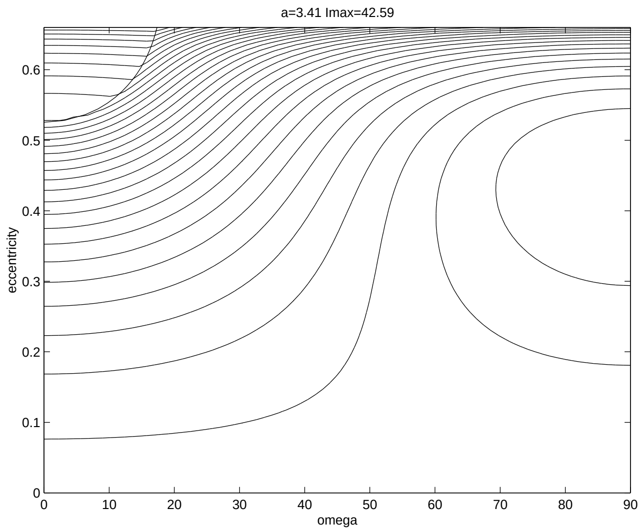

Figure 21. Level lines of the averaged perturbing function of an orbit which could cross the orbit of Jupiter. The Jupiter-crossing line is also drawn, and the level lines are non smooth along it, because of the divergence phenomenon discussed in the text.

asteroid out of the plane of the perturbing planets. Figure 20 shows the phase space of the averaged problem, for values of the initial conditions consistent with those of the asteroid *(3040) Kozai*. A large libration region is "safe" from node crossings; there are also many solution curves where a strong $e - \omega$ coupling achieves the same result, because ω can be near 0, but only for e near its minimum value.

The same mechanism protects the high inclination asteroids at the outer edge of the main belt from close approaches to Jupiter, as in Figure 21, because $\omega = \pi/2$ implies that not only the perihelion, but also the aphelion is out of the plane of the planetary orbits. When on the contrary the planet with which collision could occur has a semimajor axis a' very close to a, then the node crossings are avoided when the nodal points are at perihelion and at aphelion, that is for $\omega \simeq 0$ and $\omega \simeq \pi$, as in Figure 22 and in [13]. For very high inclination the same averaged dynamics can result in increases of the eccentricity up to values very close to 1, and then the fate of the orbit can be to encounter the surface of the Sun; however this requires a very high initial inclination, and occurs almost only to comets [2].

The behavior of the Kozai class asteroids implies that the Öpik-Kessler methods to compute probability of collision might fail, in that they give a finite probability to collisions which can not occur at all. A more flexible method to compute frequency of close approaches and probability of collision, which could take into account the actual occurrence of node crossings, was developed over many years, beginning with the basic formulas devised by [25], later extended and developed by [29], [6], and many others. The basic geometric idea is as follows. When two osculating orbits have a small nodal distance, it is possible to approximate a short span of both ellipses with their tangent lines at the respective nodal points, parametrized in such a way that the velocity at the nodal point is the same on the ellipses and on the straight lines. Then the distance, as a function of the parameters on these straight lines, is an easily computed quadratic form. From this approximate distance function it is possible to compute analytically the time span spent by the two orbits in the region where the distance is less than a given impact parameter.

The large scale tests performed in [21] show that the Öpik-Wetherill method is very effective in predicting the number of close approaches which is possible at each actual node crossing, thus it solves the problem of computing the probability of collision for the Kozai class, provided the secular evolution of the orbital elements e, I, ω is available, e.g. from numerical integrations. It is also a very effective method to detect Toro class orbits, which by definition violate the formula, by avoiding close approaches at actual node crossings. This method can not be used for low inclination and almost tangent encounters, the same difficulties found with the Kessler method.

The Figures 20, 21 and especially 22 show that the level curves of the averaged Hamiltonian $\overline{H}$ are not smooth on the node crossing lines. Indeed the integral over the torus of the variables ℓ, ℓ' is an improper one, and it is convergent because the singularity of collision is a pole of order one. On the contrary, if the averaging is applied to the right hand side of the equation of motion, the singularity of collision is a pole of order two, and the improper integral over the torus is not convergent. Thus the very meaning of the secular perturbation equations is uncertain.

Recently it has been shown [7] that the averaged Hamiltonian $\overline{H}$, although not smooth, has enough regularity –in a neighbourhood of the node crossing line– to allow a generalized definition of secular perturbation equations. This has been obtained by using the Öpik-Wetherill approximation of a short stretch of the orbits near the mutual node with straight lines. The approximate distance function d between the points on the straight lines is a quadratic form, and then the integral of $1/d$ over the torus can be computed either analytically or semi-analytically; the singularity of $1/d$ at exact node crossing is used to remove the singularity from the improper integral (Kantorovic method to compute improper integrals).

In this way it is possible to compute secular orbits even in the presence of node crossings, and thus to compute the secular frequency of circulation/libration of ω and of circulation of Ω. As a matter of principle, this

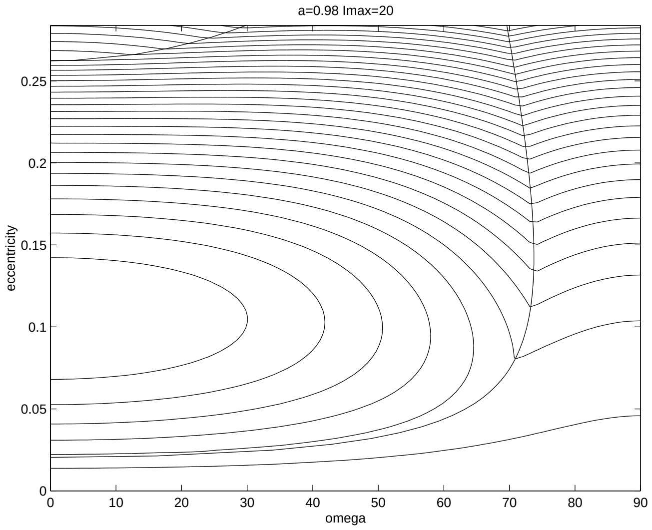

Figure 22. Averaged perturbing function of an orbit with $a = 0.98\,AU$, which could cross both the orbit of the Earth and the orbit of Venus. The safe Kozai class state is obtained for libration of ω around $0°$. Along the node crossing lines with the Earth (on the right) and with Venus (on the left) the solution curves of the averaged problem are not smooth; to define a secular frequency of libration/circulation of ω requires a suitably generalized definition of secular perturbations.

allows one to use the Kozai theory as a first integrable step in a perturbation theory, to study secular resonances, and to compute proper elements even for planet-crossing orbits; the theory developed in [7] contains explicit algorithms for this, but most of the work remains to be done.

6. THE SPACEGUARD DYNAMICAL CLASSIFICATION

The current empirical knowledge, based mostly on numerical integrations, on the dynamical behavior of Earth-crossing orbits –over time scales of a few $10,000$ to a few $100,000$ years– can be summarized in a classification of the orbit types following [20]. For most, but by no means all, of the classes some theory is available, at least as a tool to understand the qualitative features, and as a first approximation. For longer time spans (millions of years) other dynamical phenomena, such as the secular resonances, are relevant, as well as the complex effects of chaotic diffusion.

The classification uses as criteria the main features of the planet-crossing dynamics: node crossings and close approaches to the Earth, mean motion resonances with the Earth, the value of q below/above $1\,AU$, mean motion resonances and approaches with Jupiter. The logic is summarized in the block diagram of Figure 23.

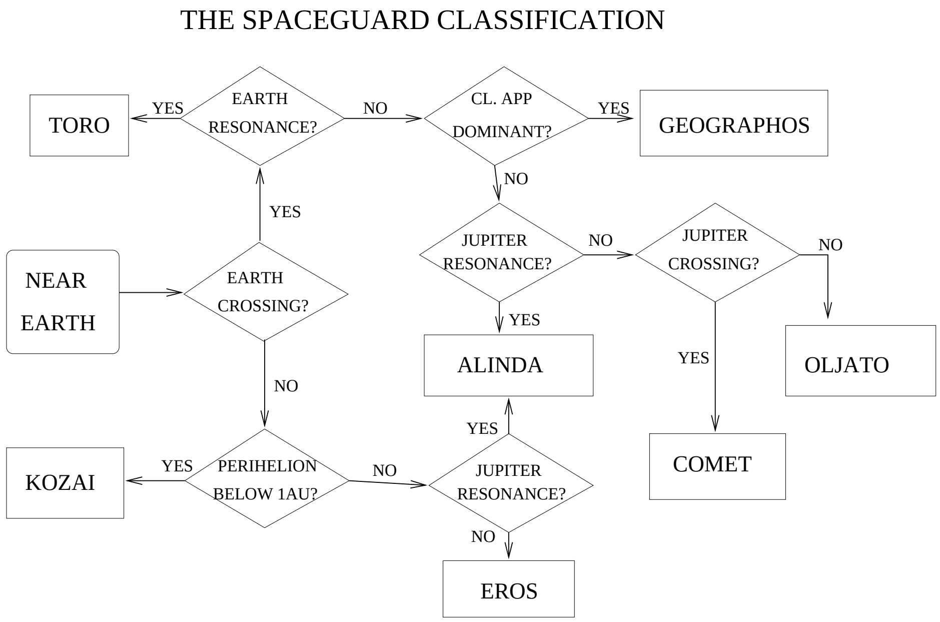

Figure 23. The SPACEGUARD dynamical classification of Near Earth Orbits: the procedure by which the orbit of a Near Earth Asteroid is assigned to a class is described as a block diagram; the questions asked in the decision boxes are explained in the text.

For each asteroid which comes "Near Earth" in some sense (say $q < 1.1\,AU$) the nodal distance is computed as a function of time; the first question is whether it has node crossings with the Earth. If there are no node crossings, but the perihelion distance is –at least for part of the time, during a period of ω– below $1\,AU$, the orbit is protected by secular perturbations and is classified as *Kozai class*. If the perihelion is above $1\,AU$, and the orbit is not involved in any major mean motion resonance with Jupiter, then it belongs to the *Eros class*; as it is the case with the namesake asteroid *(433) Eros*, it could become an Earth-crosser and have a significant probability of impacting the Earth, but only as a result of secular resonances, acting over time scales of millions of years [15], [16].

On the other hand, an asteroid which is not presently Earth-crossing could be strongly perturbed by a mean motion resonance with Jupiter; in this case, it belongs to the *Alinda class*. The behavior of the semimajor axis of an Alinda is easily identified, because the resonant perturbations by Jupiter generate large and rapid changes in a, as in the example in Figure 24. However, the reason for placing these orbits in a separate class is the very rapid secular evolution of the eccentricity shown in Figure 25; as a result, the value of q can decrease below $1\,AU$ in a very short time span, typically of the order of a few $10,000$ years.

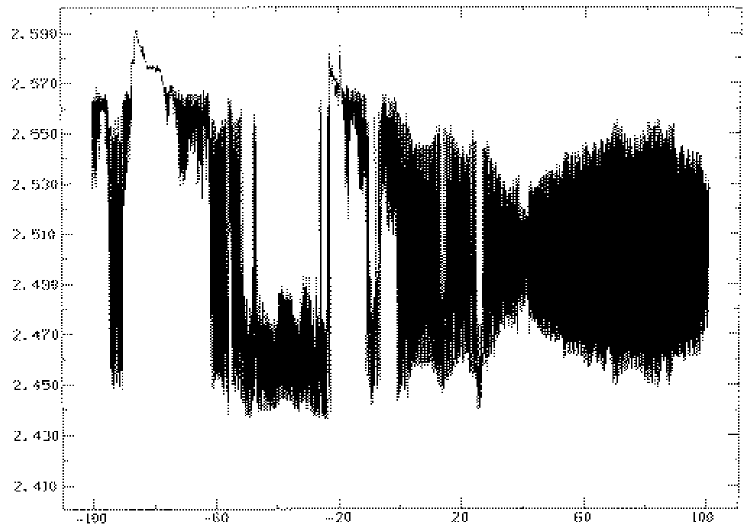

Figure 24. Semimajor axis of the asteroid *(2608) Seneca* from the output of the PROJECT SPACEGUARD numerical integration. The large changes in semimajor axis are due to a mean motion resonance with Jupiter, in this case the 3 : 1.

The superposition of the large changes in e and of the resonant changes in a results in a nodal distance which changes in a significant way both over time scales of a few hundred years –the typical period of the resonance critical arguments– and over a time scale of several thousand years –the period of the secular perturbations. This can generate the so called *super-crossing* behavior, in which node crossings take place at intervals of a few hundred years, but also a comparatively rapid change between *Apollo* orbits (with $q < 1\,AU$) and *Amor* orbits (with $q > 1\,AU$), as shown in Figure 26.

The dynamical behavior of the orbits in mean motion resonance with Jupiter has been studied with semianalytic techniques, e.g. in [22], [23], and is understood at least in qualitative terms. However, the interaction between the resonant, but long distance, perturbations by Jupiter and the short range interactions with the terrestrial planets results in a very complicated behavior of these Kirkwood gap asteroids. The evidence from numerical integrations indicates that there is a synergic effect, by which the orbital elements of the Alinda can change very rapidly, to the point that the distinction between orbits which either are or are not presently Earth-crossing is irrelevant, even over time scales of a few $10,000$ years.

Going back to the first decision box in Figure 23, if node crossings with the Earth do occur, the orbit is *Toro class* if it is sometimes locked in a mean motion resonance, and this resonance is effective to avoid (at least decrease in frequency and depth) the close approaches to our planet. If this is not the

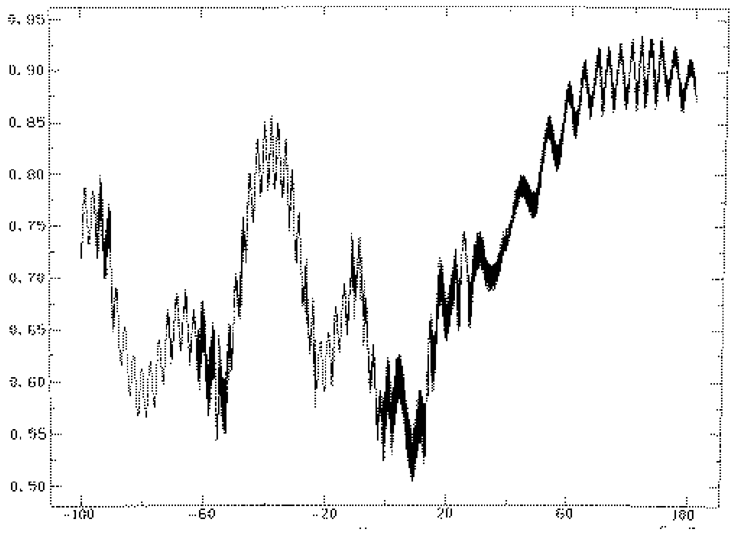

Figure 25. The eccentricity of the asteroid *(2608) Seneca* undergoes very large changes, especially when the locking in the mean motion resonance is maintained for many thousands of years, as it is the case in the positive time portion of this numerical integration. As a result, in this example the perihelion distance change from values above 1 *AU* to Mercury-crossing values.

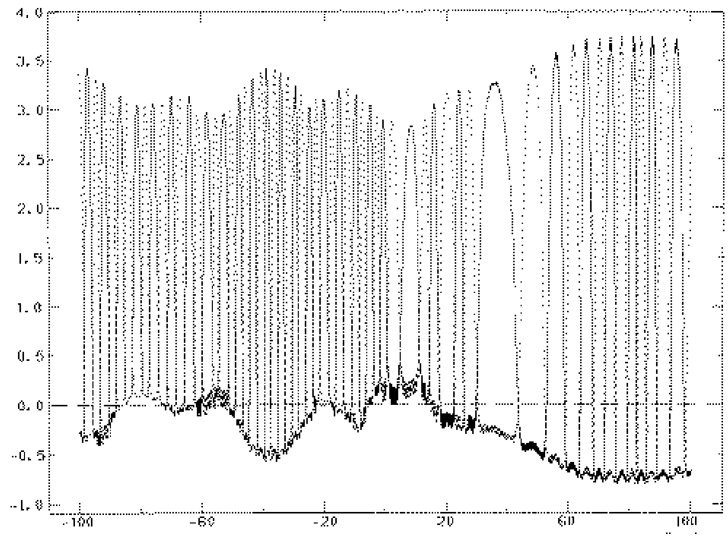

Figure 26. Nodal distance of the asteroid *(2608) Seneca* with respect to the Earth. The orbit changes between Earth-crossing and not crossing states over a few thousands of years, and also shows the peculiar super-crossing behavior, in which the nodal distance goes through zero during each cycle of the critical arguments of the 3 : 1 resonance with Jupiter, that is over a time span of a few hundred years.

case, the time series of a should be examined to decide if the contributions from short range interactions with the terrestrial planets are the dominant source of changes. A formal test based upon correlation analysis could be used, as in [21], Section 5; but in most cases visual inspection of the orbital elements plots is enough to decide. The Earth-crossing orbits for which the evolution of a is dominated by close approaches to the Earth (and in some cases to Venus) are *Geographos class*; the ones where resonant perturbations from Jupiter are dominant are again *Alinda class*.

The classification scheme described above is logical and for each class there are analytical tools to understand the dynamical behavior. Now come the cases for which the theories are still inadequate, although they can be used to describe the behavior in a qualitative sense. If the evolution of the orbital elements, in particular for the semimajor axis, is neither dominated by the close approaches nor by the mean motion resonances with Jupiter, and nevertheless node crossings take place with the terrestrial planets, the orbit is too strongly perturbed to be described by the currently available theories. If this is due to the occurrence of close approaches to Jupiter, this is a typical cometary orbit (of the so-called *Jupiter family*); the fact that objects in such orbits can be found in catalogues of asteroids should not be surprising. If it is indeed physically a comet, with a high content of volatiles, this means only that when the perihelion descends below some critical value for sublimation (depending upon the surface composition) the occurrence of an outburst is not necessary, but may depend upon the state of the surface, which in turn may depend upon the dynamical and physical history. On the other hand the same orbit could result from dynamical evolution either from an Alinda class of from a marginally unstable Trojan, and could correspond to an object with asteroidal composition.

There is still one case, corresponding to the last box on the right in Figure 23: there are orbits which are not resonant with Jupiter (at least not for a significant span of time), do not undergo orbital changes dominated by close approaches to the terrestrial planets, and are neither Jupiter crossing nor Jupiter approaching; according to the classification of the *PROJECT SPACEGUARD* they form the *Oljato class*, from the namesake asteroid *(2201) Oljato*; see [20], Figure 14. These orbits have an eccentricity very large and changing secularly, low to moderate inclination, and very frequent close approaches to the terrestrial planets; they can be described as comet-like orbits with aphelia just decoupled from Jupiter, and therefore are likely to be a very important step in the dynamical pathway of transition between Jupiter family comets and Near Earth asteroids.

The problem is, these are the orbits we are not yet able to model analytically in all their complexity, because the combined effects of secular resonances and close approaches to the terrestrial planets play an essential rôle in the evolution of the Oljato class orbits. Theories are available for secular resonances, especially the most important one, resulting from the argument of perihelion $\varpi = \omega + \Omega$ of the asteroid revolving around the Sun with the same average rate as the one of Saturn [24], but these theories are not consistently applicable when close approaches with the Earth can take place; numerical experiment confirm the existence of the secular resonances even where node crossing can occur [5]. In the near-Earth and near-Mars region other secular resonances, locking the asteroid perihelium and node to the ones of the terrestrial planets, are known to occur [14], [16], but the theory exists so far only for low eccentricities and non-crossing orbits. The Oljato class orbits are often found in the Taurid region [28]; large scale changes in the orbital elements, resulting from resonances, provide the entrance path into this region, with orbits coming both from the asteroid main belt and from the Jupiter family of comets, and the exit path: collision with the Sun [4], hyperbolic expulsion after a close approach to Jupiter, and less frequently collision with one of the terrestrial planets.

I would like to conclude by saying that the dynamical classification of [20] is an useful tool to understand the dynamics of the Earth-approaching objects, but it needs to be stressed that all these are strongly chaotic orbits, and that all the orbital elements can undergo large relative changes, as a result of either deep close approaches or the action of some resonance. Thus transitions can occur (see [20], Figure 16), and indeed often occur, between the dynamical classes. The Toro and Alinda classes are especially unstable, with typical residence times in the class $< 100,000$ years. As an example, transitions can occur between the Alinda, comet-like and Oljato classes; that is, an asteroid coming from some Kirkwood gap can be injected into a comet-like orbit as well as ending up as a meteorite.

Acknowledgements

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